Lévy–Student processes for a Stochastic model of Beam Halos

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1 Outline

The charged particle beams dynamics and their possible halos are here described in terms of stochastic processes.

To have time reversal invariance a dynamics must be added: the Wiener noise $W(t)$ being on the position equation, the position $Q(t)$ is Markovian, but not derivable; hence we drop the momentum equation, we work in a configuration space, and the dynamics is introduced by a stochastic variational principle where $v_+(r, t)$ is a new dynamical variable.

This scheme, the Stochastic mechanics (SM), is known for its application to classical stochastic models for Quantum Mechanics, but is suitable for a large number of other systems.
In SM the Lagrange equations are equivalent to a Schrödinger–like (S–ℓ) equation: we will speak of quantum–like (Q-ℓ) systems. A new role for the beam dynamics can be played by non–Gaussian Lévy distributions. Their today’s popularity is mainly confined to the stable laws. We instead introduce a family of non–Gaussian Lévy laws which are infinitely divisible but not stable: the generalized Student laws:

- stable non–Gaussian laws – but not Student laws – always have divergent variances;
- Student laws can approximate the Gaussian laws;
- the i.d. laws is all that is required to build the Lévy processes used to represent the evolution of our particle beam.
The Student laws are here used in two ways:

- in the framework of the traditional SM, with randomness supplied by a *Gaussian Wiener noise*, we study the self–consistent potentials which can produce a *Student distribution as stationary transverse distribution* of a particle beam, and we focus our attention on the increase of the probability of finding the particles far away from the beam core.

- we define a *Lévy–Student process*, and we show that these processes can help to explain how a particle can be expelled from the bunch by means of some kind of hard collision. In fact the trajectories of our Lévy–Student process show the *typical jumps of the non–Gaussian Lévy processes*: a feature that we propose to use as a model for the halo formation.
2 Stochastic beam dynamics

The position $Q(t)$ of a representative particle in the beam is a process ruled by the Itô stochastic differential equation (SDE)

$$dQ(t) = v(+) (Q(t), t) \, dt + \sqrt{D} \, dW(t),$$

(1)

- $v(+) (r, t)$ is the forward velocity
- $dW(t)$ is the increment process of a standard Wiener noise
- the diffusion coefficient $D$ is constant, and the action $\alpha = 2mD$ will be later connected to the emittance of the beam.

To add a dynamics we introduce a stochastic least action principle and we get a Nelson process.
\( \rho(\mathbf{r}, t) \) is the pdf of \( \mathbf{Q}(t) \): define \textit{backward velocity, current and osmotic velocities}

\[
\mathbf{v}(-) = \mathbf{v}(+) - 2D \frac{\nabla \rho}{\rho}, \quad \mathbf{v} = \frac{\mathbf{v}(+) + \mathbf{v}(-)}{2}, \quad \mathbf{u} = \frac{\mathbf{v}(+) - \mathbf{v}(-)}{2}
\] (2)

From the stochastic least action principle:

• the current velocity is \textit{irrotational}

\[
m \mathbf{v}(\mathbf{r}, t) = \nabla S(\mathbf{r}, t), \quad (3)
\]

• the \textit{Lagrange equations of motion} for \( \rho \) and \( S \) are

\[
\partial_t \rho = -\frac{1}{m} \nabla \cdot (\rho \nabla S)
\] (4)

\[
\partial_t S = -\frac{1}{2m} \nabla S^2 + 2mD^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} - V(\mathbf{r}, t)
\] (5)
• The system is \textit{time–reversal invariant}.

• The forward velocity $v_{(+)}(r, t)$ is not given \textit{a priori}, but it is
dynamically determined by the evolution equation (5).

• With the representation
  \begin{equation}
  \Psi(r, t) = \sqrt{\rho(r, t)} e^{iS(r, t)/\alpha}, \quad \alpha = 2mD
  \end{equation}

the coupled equations (4) and (5) become a single linear
equation of the form of the Schrödinger equation, with the
Planck action constant replaced by $\alpha$:

\begin{equation}
  i\alpha \partial_t \Psi = -\frac{\alpha^2}{2m} \nabla^2 \psi + V \Psi.
\end{equation}

We will refer to it as a \textit{Schrödinger–like equation}. 
3 Self-consistent equations

In the SM scheme $|\Psi(r, t)|^2$ is the pdf of a Nelson process:

- When $N$–particles are a pure ensemble, $N|\Psi(r, t)|^2 d^3r$ is the number of particles in a small neighborhood of $r$.

- Our $N$ particles are not a pure ensemble due to their mutual e.m. interaction: in a mean field approximation we will take into account the so called space charge effects. We will couple the S–$\ell$ equation with the Maxwell equations of both the external and the space charge e.m. fields, and we will get a non linear system of coupled differential equations.
The space charge and current densities are

\[
\rho_{sc}(\mathbf{r}, t) = Nq_0|\Psi(\mathbf{r}, t)|^2,
\]
\[
\mathbf{j}_{sc}(\mathbf{r}, t) = Nq_0 \frac{\alpha}{m} \Im\{\Psi^*(\mathbf{r}, t) \nabla \Psi(\mathbf{r}, t)\}.
\]

The e.m. potentials \((\mathbf{A}_{sc}, \Phi_{sc})\) and \(\Psi\) then obey the system

\[
0 = \nabla \cdot \mathbf{A}_{sc}(\mathbf{r}, t) + \frac{1}{c^2} \partial_t \Phi_{sc}(\mathbf{r}, t)
\]
\[
\mu_0 \mathbf{j}_{sc}(\mathbf{r}, t) = \nabla^2 \mathbf{A}_{sc}(\mathbf{r}, t) - \frac{1}{c^2} \partial^2_t \mathbf{A}_{sc}(\mathbf{r}, t)
\]
\[
\frac{\rho_{sc}(\mathbf{r}, t)}{\varepsilon_0} = \nabla^2 \Phi_{sc}(\mathbf{r}, t) - \frac{1}{c^2} \partial^2_t \Phi_{sc}(\mathbf{r}, t)
\]
\[
i \frac{\alpha}{2m} \partial_t \Psi = \left[i\alpha \nabla - \frac{q_0}{c} (\mathbf{A}_{sc} + \mathbf{A}_e)\right]^2 \Psi + q_0(\Phi_{sc} + \Phi_e)\Psi
\]
For stationary wave functions

\[ \Psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-iEt/\alpha}, \quad V_e(\mathbf{r}) = q_0 \Phi_e(\mathbf{r}), \quad V_{sc}(\mathbf{r}) = q_0 \Phi_{sc}(\mathbf{r}), \]

For cylindrical symmetry with constant \( p_z \) and beam length \( L \)

\[ \psi(\mathbf{r}) = \chi(r, \varphi) \frac{e^{ip_z z/\alpha}}{\sqrt{L}}, \quad p_z = \frac{2k\pi\alpha}{L}, \quad k = 0, \pm 1, \ldots \quad (10) \]

For \( N = N/L, \ E_T = E - p_z^2/2m, \ \chi(r, \varphi) = u(r)\Phi(\varphi), \) zero angular momentum, and dimensionless quantities \((\eta, \lambda \text{ d.c.})\)

\[ s = \frac{r}{\lambda}, \quad \beta = \frac{E_T}{\eta}, \quad \xi = \frac{Nq_0^2}{2\pi\epsilon_0\eta} \quad \text{(perveance)} \]

\[ w(s) = \lambda u(\lambda s) \]

\[ v(s) = \frac{V_{sc}(\lambda s)}{\eta}, \quad v_e(s) = \frac{V_e(\lambda s)}{\eta} \]
We get the radial, stationary, cylindrical, dimensionless equations

\[ s w''(s) + w'(s) = [v_e(s) + v(s) - \beta] s w(s) \]  \quad (11)

\[ s v''(s) + v'(s) = -\xi s w^2(s) \]  \quad (12)

We can look at our equations in two different ways:

- \( v_e \) is a given external potential: solve the system for \( w \) and \( v \). No simple analytical solution – playing the role of the Kapchinskij–Vladimirskij distribution – is available.

- \( w \) is a given radial distribution: solve the system for \( v_e \) and \( v \). Analytical solutions are available.

We adopted the first in previous papers where we numerically solved the equations (11) and (12); here we will elaborate a few ideas about the second one.
Poisson equation (12) with $v(0^+) = v'(0^+) = 0$ gives the space charge potential

$$v(s) = -\xi \int_0^s \frac{dy}{y} \int_0^y x w^2(x) \, dx$$  \hspace{1cm} (13)

From the first equation (11) we obtain also the external potential

$$v_e(s) = v_0(s) + \xi \int_0^s \frac{dy}{y} \int_0^y x w^2(x) \, dx ,$$ \hspace{1cm} (14)

$$v_0(s) = \frac{w''(s)}{w(s)} + \frac{1}{s} \frac{w'(s)}{w(s)} + \beta$$ \hspace{1cm} (15)

where $v_0(s)$ is the zero perveance potential that we get without space charge (perveance $\xi = 0$).
4 Self–consistent potentials

First take the radial *ground state of the harmonic oscillator* with zero perveance

$$u_0(r) = \frac{e^{-r^2/\sigma^2}}{\sigma}$$ (16)

with $E_T = \alpha \omega$ and

$$V_e(r) = \frac{m\omega^2}{2} r^2 = \frac{\alpha^2}{8m\sigma^4} r^2, \quad \sigma^2 = \frac{\alpha}{2m\omega}$$ (17)

Dimensionless representation ($\eta = \alpha \omega / 2, \lambda = \sigma \sqrt{2}$):

$$w(s) = \sqrt{2} e^{-s^2/2}, \quad \beta = 2, \quad v_e(s) = s^2$$ (18)
The **external and the space charge potentials** that produce (18) as stationary wave function

\[ v(s) = -\frac{\xi}{2} \left[ \log(s^2) + C - \text{Ei}(-s^2) \right] \]  
(19)

\[ v_0(s) = s^2 \]  
(20)

\[ v_e(s) = s^2 + \frac{\xi}{2} \left[ \log(s^2) + C - \text{Ei}(-s^2) \right] \]  
(21)

where \( C \approx 0.577 \) is the Euler constant and

\[ \text{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} \, dt , \quad x < 0 \]

is the exponential–integral function (see Figure 1)
Figure 1: The dimensionless potentials $v(s)$ (thin line), $v_0(s) = s^2$ (dashed line) and $v_e(s)$ (thick line) for $\xi = 20$. With this external potential the self–consistent wave function coincides with that of a simple harmonic oscillator for zero perveance (18).
If a halo is produced by \textit{large deviations from the beam axis}, alternatively suppose that the \textbf{the stationary transverse distributions are non–Gaussian}: take this family of univariate, two–parameters probability \textit{laws} $\Sigma(\nu, a^2)$ with pdf’s

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\nu}{2}\right)} \frac{a^\nu}{(x^2 + a^2)^{\frac{\nu+1}{2}}} , \quad \nu > 0 \quad (22)$$

with mode and median in $x = 0$, two flexes in $x = \pm a/\sqrt{\nu + 2}$. $a$ is a scale parameter, while $\nu$ rules the \textit{power decay of the tails}: for large $x$ the tails go as $x^{-(\nu+1)}$. Compare with a Gauss law $\mathcal{N}(0, \sigma^2)$ (Figure 2): when $\nu$ grows the difference between the two pdf’s becomes smaller.
Figure 2: The Gauss pdf $\mathcal{N}(0, 1)$ (dashed line) compared with the $\Sigma(2, 2)$ (thick line) and the $\Sigma(10, 12)$ (thin line). The flexes of the three curves coincide. Apparently the tails of the $\Sigma$ laws are much longer.
Since $\Sigma(n, n)$ with $n = 1, 2, \ldots$ are the classical $t$–Student laws, we call $\Sigma(\nu, a^2)$ generalized Student laws. They have finite variance only when $\nu > 2$

$$\sigma^2 = \frac{a^2}{\nu - 2}. \quad (23)$$

Then for $\nu > 2$, $\Sigma(\nu, (\nu - 2)\sigma^2)$ has variance $\sigma^2$.

The circularly symmetric, bivariate Student laws $\Sigma_2(\nu, a^2)$ are

$$f(x, y) = \frac{\nu}{2\pi} \frac{a^\nu}{(x^2 + y^2 + a^2)^{\nu+2}/2}. \quad (24)$$

with non–correlated (but not independent) marginals $\Sigma(\nu, a^2)$. 
The beam distribution with finite transverse variance $\sigma^2$ is

$$\rho(r, \varphi, z) = r \frac{\nu}{2\pi L} \frac{[(\nu - 2)\sigma^2]^{\frac{\nu}{2}}}{[r^2 + (\nu - 2)\sigma^2]^{\frac{\nu+2}{2}}} H \left( \frac{L}{2} - |z| \right)$$

and the radial, dimensionless distribution for $\nu > 2$ is

$$w^2(s) = \frac{2\nu}{\nu - 2} \frac{1}{(1 + z^2)^{\frac{\nu+2}{2}}} , \quad z = \frac{s\sqrt{2}}{\sqrt{\nu - 2}} \tag{25}$$

with dimensional constants

$$\eta = \frac{\alpha^2}{4m\sigma^2} , \quad \lambda = \sigma\sqrt{2} \tag{26}$$
The potentials with (25) as radial stationary distribution are

\[ v(s) = -\frac{\xi}{2} \left[ \frac{2z^{-\nu}}{\nu} \, _2F_1 \left( \frac{\nu}{2}, \frac{\nu}{2} ; \frac{\nu + 2}{2} ; -\frac{1}{z^2} \right) \right. \]

\[ \left. + \log z^2 + C + \psi \left( \frac{\nu}{2} \right) \right] \quad (27) \]

\[ v_0(s) = \frac{\nu + 2}{\nu - 2} \frac{z^2(4z^2 + \nu + 10)}{2(1 + z^2)^2}, \quad z = \frac{s\sqrt{2}}{\sqrt{\nu - 2}} \quad (28) \]

\[ v_e(s) = v_0(s) - v(s), \quad \beta = 2 + \frac{8}{\nu - 2} \quad (29) \]

where \(_2F_1(a, b; c; w)\) is a hypergeometric function and \(\psi(w) = \Gamma'(w)/\Gamma(w)\) is the logarithmic derivative of the Euler Gamma function (digamma function).

\(v_0(s)\) is the control potential for zero perveance (Figure 3)
Figure 3: The zero perveance potential $v_0(s)$ (28) for a Student transverse distribution $\Sigma_2(22, 20\sigma^2)$. Also displayed: $\beta = 2.4$ (the limit value of $v_0$ for large $s$, thin line) and the behaviors for small and large $s$ (dashed lines).
Compare with the potentials of a Gaussian distribution:

- space charge potentials $v(s)$ (Figure 4) are similar:
  - for $s \to +\infty$ both behave as $-\xi \log s$.

- zero perveance potentials $v_0(s)$ (Figure 5) look different for large $s$, but the difference fades away for large $\nu$:
  - in the Gauss case the potential diverges as $s^2$
  - in the Student case it goes to $\beta$ as $s^{-2}$

- total external potentials $v_e(s) = v_0(s) - v(s)$ (Figure 6)

Even if the potential near the beam axis is harmonic, deviations from this behavior in a region removed form the core can produce a deformation of the distribution from the gaussian to the Student.
Figure 4: The space charge potentials $v(s)$ respectively for a Student (solid line) distribution $\Sigma_2(22, 20\sigma^2)$, and for a Gauss (dashed line) distribution. Dimensionless perveance $\xi = 20$. 
Figure 5: The zero perveance potential $v_0(s)$ (28) of a Student $\Sigma_2(22, 20\sigma^2)$ (solid line; see FIG. 3) compared with that of a Gauss distribution (dashed line) with the same behavior near the beam axis.
Figure 6: The total external potential $v_e(s)$ (29) that should be applied to get a stationary Student transverse distribution $\Sigma_2(22, 20\sigma^2)$ (solid line), compared with that (21) needed for a Gauss distribution (dashed line).
\( P(c) \) probability of being beyond distance \( c\sigma \) from the beam axis:

- **Gauss case**
  \[
P(c) = e^{-c^2/2}, \quad P(10) \approx 1.9 \times 10^{-22}
\]

- **Student case**
  \[
P_\nu(c) = \left(1 + \frac{c^2}{\nu - 2}\right)^{-\nu/2}
\]
  \[
\begin{align*}
P_{10}(10) & \approx 2.2 \times 10^{-6} \\
P_{22}(10) & \approx 2.8 \times 10^{-9}
\end{align*}
\]

For \( \mathcal{N} = 10^{11} \) particle per meter of beam, we find beyond 10\( \sigma \)

- practically no particle in the Gaussian case
- between 10\(^3\) and 10\(^5\) in the Student case

We got the same numbers in the numerical solutions for \( \xi = 20 \).
5 Lévy–Student processes

The Student laws \( \Sigma(\nu, a^2) \) are a family of Lévy \textit{infinitely divisible (i.d.) laws}. Present interest about non–Gaussian Lévy laws (from physics to finance) is mostly confined to the \textit{stable laws}: a sub–family of the i.d. laws.

- The i.d. laws constitute the more general form of possible limit laws for the generalized \textit{Central Limit Theorem}.

- The i.d. laws constitute the class of all the laws of the increments for every stationary, stochastically continuous, independent increments process (\textit{Lévy process}).
Non–Gaussian Lévy process have trajectories with moving discontinuities (e.g. compound Poisson process): a possible model for the relatively rare escape of particles from the beam core.

We will limit ourselves to 1–DIM systems.

*Characteristic function* (ch.f.) of a random variable (r.v.) $X$

$$\varphi(\kappa) = \mathbb{E}(e^{i\kappa X})$$

The law $\mathcal{L}$ of the *sum of $n$ independent r.v.’s* is

$$\varphi(\kappa) = \varphi_1(\kappa) \cdot \ldots \cdot \varphi_n(\kappa) \quad (30)$$

A law $\mathcal{L}$ is *decomposed* in the laws $\mathcal{L}_1, \ldots, \mathcal{L}_n$ when (30) holds.
A law $\mathcal{L}$ is i.d. when for every $n$ there is a law $\mathcal{L}_n$ such that
\[ \varphi = \varphi^n \]
i.e. when the r.v. $X$ can always be decomposed in the sum of $n$
inddependent r.v.’s all with the same law $\mathcal{L}_n$.

The laws $\mathcal{L}_n$ are not in general of the same type as $\mathcal{L}$. Two
laws are of the same type when they differ by a centering and a
rescaling: $e^{iak} \varphi(bk)$ for every $a$ and $b > 0$.

A law $\mathcal{L}$ is stable when it is i.d. and the component laws are of
the same type as $\mathcal{L}$: for every $b, b' > 0$, exist $a$ and $c$ such that
\[ \varphi(c_k) = e^{iak} \varphi(bk) \varphi(b'_k) . \]

Gauss and Cauchy laws are stable; Poisson laws are only i.d.
Central Limit Problem: the family of i.d. laws coincides with the family of the limit laws of the consecutive sums

\[ S_n = \sum_{k=1}^{n} X_{n,k} \]  

with \( X_{n,1}, \ldots, X_{n,n} \) independent for every \( n \). The family of stable laws coincides with the family of the limit laws of the normed (centered and rescaled) sums

\[ S_n = \frac{S^*_n}{a_n} - b_n, \quad S^*_n = \sum_{k=1}^{n} X_k \]

with \( X_k \) identically distributed: a particular case of (31) for

\[ X_{n,k} = \frac{X_k}{a_n} - \frac{b_n}{n} \]
Lévy–Khintchin formula: gives the ch.f.’s of i.d. and stable laws. For stable laws these ch.f.’s are explicitly known in terms of elementary functions. For i.d. laws the ch.f.’s are given through an integral containing a (Lévy function) $L(x)$ associated to every particular law. In most cases the Lévy functions are not known.

Decomposable processes: Markov processes with independent increments: the laws of the increments $\Delta X(t)$ must be i.d. laws.

Lévy process: a decomposable process $X(t)$ stationary and stochastically continuous.

A Lévy process can have moving, as opposed to fixed discontinuities (e.g. Poisson process). Only the Gaussian Lévy processes (e.g. Wiener process) are pathwise continuous: almost every sample path is everywhere continuous).
If $\varphi(\kappa)$ is i.d. and $T$ is a time constant, then $[\varphi(\kappa)]^{\Delta t/T}$ is the ch.f. of $\Delta X(t)$ of a Lévy process with stationary transition pdf

$$p(x, t|y, s) = \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} e^{ik(x-y)}[\varphi(\kappa)]^{\frac{t-s}{T}} d\kappa$$  \hspace{1cm} (33)$$

Almost all trajectories are continuous with the exception of a countable set of moving jumps. If $L_t(x)$ is the Lévy–Khintchin function of the i.d. law of the increment $X(s + t) - X(s)$, and $\nu_t(x)$ is the random number of the jumps in $[s, s + t)$ of height in absolute value larger than $x > 0$, then

$$|L_t(x)| = \mathbb{E}(\nu_t(x))$$

namely: the Lévy–Khintchin function is a measure of the frequency and height of the trajectory jumps.
The ch.f. of a Student law $\Sigma(\nu, a^2)$ is

$$\varphi(\kappa) = 2 \frac{|a\kappa|^\nu K_{\nu/2}(|a\kappa|)}{2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right)}$$

(34)

where $K_\alpha(z)$ is a modified Bessel function (see Figure 7). They are i.i.d. but in general are not stable.

- all Student laws with $\nu > 2$ have a finite variance, while no stable, non-Gaussian law can have it: there is no need to resort to truncated Lévy distributions;

- stable, non-Gaussian laws decay as $|x|^{-\alpha-1}$ with $\alpha < 2$, while the Student laws go as $|x|^{-\nu-1}$ with $\nu > 0$; this allows the Student laws to approximate the Gaussian behavior as well as we want.
Figure 7: Typical ch.f. of a Student law $\Sigma(2, 2)$ (solid line) compared with a standard Gauss law $\mathcal{N}(0, 1)$ (dashed line).
A Lévy process defined by the ch.f. (34) will be called in the following a Lévy–Student process. Its transition pdf is

\[
p(x, t|y, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\kappa(x-y)} \left[ \frac{2|a\kappa|^\frac{\nu}{2} K_{\frac{\nu}{2}}(|a\kappa|)}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \right]^{\frac{t-s}{T}} d\kappa
\]  

(35)

In principle (35) is enough to calculate everything of our process, but in practice this integral must be treated numerically.

For \(t - s = T\) (35) coincides with the pdf of a Student \(\Sigma(\nu, a^2)\): we can then produce sample trajectory simulations by taking \(T\) as the time step, since the increments are exactly Student distributed when observed at the (arbitrary) time scale \(T\).
We produce a *simplified model* which simulates the solutions of the following two SDE’s

\[ dX(t) = v(X(t), t) \, dt + dW(t) \]  \hspace{1cm} (36)

\[ dY(t) = v(Y(t), t) \, dt + dS(t) \]  \hspace{1cm} (37)

\( W(t) \) is a Wiener process

\( S(t) \) is a Lévy-Student process

\( v(x, t) \) is \( t \)-independent, and is (for given \( b > 0 \) and \( q > 0 \))

\[ v(x) = -bx \, H(q - |x|) \]

where \( H \) is the Heaviside function.

This flux will attract the trajectory toward the origin when \( |x| \leq q \), and will allow the movement to be completely free for \( |x| > q \).
Figure 8: The pdf’s of the increments for the Gaussian processes (dashed line; $\sigma \simeq 0.53$) and for the Lévy–Student process with law $\Sigma(4,1)$ (solid line; $\sigma \simeq 0.71$). The parameters give to the two pdf’s the same modal values and similar shapes.
Laws of the increments

- $\Delta X(t)$ of (36) has a Gaussian distribution with $\sigma = 0.53$
- $\Delta Y(t)$ of (37) has a Student distribution $\Sigma(4, 1)$

Their pdf’s look not very different. That notwithstanding the process $Y(t)$ differs in several respects from $X(t)$.

Suppose that the velocity field has $b = 0.35$ and $q = 10$. The following Figures display the typical trajectories of a $10^4$ steps solution $X(t)$ and $Y(t)$ respectively of (36) and (37).

In the Gaussian case with $\sigma$ small w.r.t $q$ the trajectories always stay inside the beam core, and the process is essentially an Ornstein–Uhlenbeck position process
In the **Student case** the trajectories:

- show a wider *dispersion* and a few larger spikes
- have the propensity to make *occasional excursions* far away from the beam core
- and seldom they also *definitely drift away* from the core

The trajectories of a non–Gaussian Lévy process are only stochastically, and not pathwise continuous: they contain occasional jumps. *The frequency and the size of these jumps can be fine tuned* by suitably choosing the values of the parameters of the law $\Sigma(\nu, a^2)$ of the increments. This feature of a Lévy–Student process suggests to adopt this model to **describe the rare escape of particles away from the beam core.**
Figure 9: Typical trajectory of a stationary, Gaussian (Ornstein–Uhlenbeck) process. To compare it with the Student trajectory, the vertical scale has been set equal to that of Figure 10.
Figure 10: Typical trajectory of a stationary, Student process ($\nu = 4$ and $a = 1$).
Figure 11: Occasional trajectory of a stationary, Student process with a temporary excursion out of the core ($\nu = 4$).
Figure 12: Rare, but possible trajectory of a stationary, Student process: here the particle definitely drifts away from the core $(\nu = 4)$. 
6 Challenges ahead

- Find the Lévy–Khintchin functions of the Student laws to fine tune the frequency and the size of the jumps.
- Find the form of the increment laws at different time scales.
- Find the integro-differential form of the Chapman–Kolmogorov equation to discuss the time evolution of the process.
- Add a dynamics to have controlled diffusions: namely to build a generalized SM for the Lévy–Student processes.
- Search for empirical or numerical evidence to support the hypothesis that the path increments of a beam are in fact distributed according to a Student law.