Rayleigh-Taylor Instability in solids

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The LAPLAS experiment

- LAPLAS consists in the low entropy implosion of a thick cylindrical shell of heavy metal that compress a material sample in the axial region.

- Conditions for the development of Rayleigh-Taylor instability arise during acceleration and deceleration phases.

- The pusher remains in solid state and it retains its elastic - plastic properties. The absorber is in liquid state.
Rayleigh-Taylor Instability (RTI) in solids

- The problem is relevant to other HEDP experiments involving the cylindrical implosion of liners driven by intense electrical currents, and the acceleration of metallic plates by high explosives or by laser beams.

- In geophysics RTI in solids plays a central role in the thickening of the lithosphere.

- In astrophysics RTI in elastic solids may be at the origin of gamma burst produced by accreting neutron stars with magnetosphere.

- The non-linear character of the constitutive properties makes the problem very difficult.

- We have developed a simple but still accurate physical model that allows for dealing with complex situations.
Incompressible Rayleigh-Taylor instability: the simplest case

- Rayleigh-Taylor instability occurs whenever a medium with density \( \rho_1 \) pushes and accelerates another medium with higher density \( \rho_2 \) (\( \rho_2 > \rho_1 \)). In the reference frame of the interface we have a constant gravitational field \( g \).

- The interface has a small sinusoidal perturbation of amplitude \( \xi(x) \) and wavelength \( \lambda \) (\( k\xi \ll 1, k = 2\pi / \lambda \)). It creates a pressure difference \( \Delta p \) on the interface that tends to further deform it:

\[
\Delta p = (\rho_2 - \rho_1) g \xi
\]

- Since motion affects the fluid within a distance \( k^{-1} \) from the interface, we can get the interface equation of motion by using the Newton second law:

\[
m \ddot{\xi} = \Delta p A; \quad \Delta p = (\rho_2 - \rho_1) g \xi; \quad m = m_1 + m_2 = \rho_1 \frac{A}{k} + \rho_2 \frac{A}{k}
\]

This simple model yields the classical result

\[
\frac{(\rho_2 + \rho_1)}{k} \ddot{\xi} = (\rho_2 - \rho_1) g \xi \quad \text{or} \quad \ddot{\xi} = A_T g \xi; \quad A_T = \frac{(\rho_2 + \rho_1)}{(\rho_2 - \rho_1)}
\]
Rayleigh-Taylor instability

- If other forces $f_i$ are present, due to viscosity, surface tension, elasticity, plasticity, ablation, accretion, etc, they must be added to the equation of motion (Piriz et al., AJP 2006):

$$\frac{d}{dt}[(m_1 + m_2)\ddot{\xi}] = (\rho_2 - \rho_1)g\ddot{\xi}A + \sum_i f_i$$

- If we consider a solid plate of thickness $h$ accelerated by a rippled pressure:

$$p = p_0(1 + \frac{\xi_0}{h}e^{ky}\sin kx),$$

then, the evolution of the interface is described by the following equation:

$$\frac{\rho}{k} \ddot{\eta} = \rho g(\eta + \eta_0) - S_{yy}$$

$$\eta = \ddot{\xi}(t)e^{ky}\sin kx$$

$S_{yy}$ is the normal component of the deviatoric part of the stress tensor $\sigma_{ij} = -p\delta_{ij} + S_{ij}$. 
Elastic-plastic constitutive model

For calculating $S_{ij}$ for an elastic plastic solid, we assume a Prandtl-Reuss rule with the Levi- von Mises yield stress criterion:

$$\dot{S}_{ij} = 2GD_{ij}, \quad \text{if} \quad S_{ij}D_{ij} < 0 \quad \text{or} \quad S_{ij}S_{ij} < \frac{2}{3}Y^2$$

$$\frac{\dot{S}_{ij}}{2G} + S_{ij} \frac{S_{j}D_{ij}}{S_{ij}S_{ij}} = D_{ij}, \quad \text{if} \quad S_{ij}D_{ij} > 0 \quad \text{and} \quad S_{ij}S_{ij} = \frac{2}{3}Y^2$$

$G$ is the shear modulus

$Y$ is the yield strength

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \dot{e}_{ij}$$
Equation of motion of the interface

- With our velocity field \( v_y = \dot{\eta} = \dot{\xi}(t)e^{ky}\sin kx \); \( v_x = \dot{\xi}(t)e^{ky}\cos kx \), and for a solid/vacuum interface, the equation of motion of the interface turns out (with stress-free initial conditions):

\[
\frac{\rho}{k} \ddot{\xi} = \frac{\rho g (\xi + \xi_0) - S_{yy}}{4\pi G} \quad ; \quad S_{yy} = \begin{cases} 
2kG\xi & \text{for } \xi \leq \xi_p \\
\frac{\alpha Y}{\sqrt{3}} & \text{for } \xi \geq \xi_p
\end{cases}
\]

\[
\alpha = e^{k\xi_p} \equiv 3 \quad \xi_p = \frac{\alpha Y}{2\sqrt{3}kG}
\]

- Introducing the following dimensionless magnitudes:

\[
z = \frac{\xi}{\xi_0} \quad ; \quad \tau = t\sqrt{kg} \quad ; \quad \hat{\lambda} = \frac{\rho g \lambda}{4\pi G} \quad ; \quad \hat{\xi} = \frac{\rho g \xi_0}{\sqrt{3}Y}
\]

the equation of motion reads:

\[
\ddot{z} = \begin{cases} 
\frac{z(1 - \hat{\lambda}^{-1}) + 1}{z + 1 - \hat{\xi}^{-1}} & \text{for } z \leq z_p \\
\frac{\hat{\lambda}}{z} & \text{for } z \geq z_p
\end{cases}
\]

\[
z(p) = 0 \quad ; \quad \dot{z}(0) = 0
\]

This equation can be integrated twice to yield an analytic expression for \( z(\tau) \).
Instability threshold for RTI in EP solids

To obtain a criterion of stability is enough to get the first integral:

\[
\ddot{z}^2 = \begin{cases} 
   z^2(1 - \hat{\lambda}^{-1}) + 2z & \text{for } z \leq z_p \\
   (z + 1 - \hat{\xi}^{-1})^2 - \frac{C^2}{\hat{\xi}^2} & \text{for } z \geq z_p 
\end{cases}
\]

\( C^2 = (1 - \hat{\xi})^2 - \hat{\lambda} \)

- The interface will be stable if \( z(\tau) \) has a maximum at \( \tau = \tau_m \). Then, when \( \dot{z}(\tau_m) = 0 \) it must be \( \ddot{z}(\tau_m) < 0 \). If at that moment it is \( \ddot{z}(\tau_m) \geq 0 \), we have an inflection point at \( \tau_m \) and the interface will be unstable. Therefore, we get the following stability condition:

\[ \hat{\xi} < \hat{\xi}_{TH} = 1 - \sqrt{\hat{\lambda}} \]

In dimensional magnitudes, the instability threshold reads:

\[ \left( \frac{\rho g \hat{\xi}}{\sqrt{3Y}} \right)_{TH} = 1 - \sqrt{\frac{\rho g \hat{\lambda}}{4 \pi G}} \]

The marginal stability conditions can only be satisfied if \( z \geq z_p \).
Elastic-plastic transition

- We can also get the condition for the transition from elastic to plastic regime by requiring that the maximum amplitude achieved in the perfectly elastic case be equal to the amplitude necessary to reach the elastic limit:

\[ \frac{\rho g \xi_0}{\sqrt{3Y}} \bigg|_{\text{EP}} = \frac{1}{2} \left( 1 - \frac{\rho g \lambda}{4\pi G} \right) \]

- Stability in EP solids requires both, small initial perturbation amplitudes and short wavelengths.


- Dots are the result of extensive 2D numerical simulations with the finite element code ABAQUS.
A connection with the Drucker criterion

- We can calculate the maximum perturbation amplitude
  \[ z_m^{TH} = z(\tau_m^{TH}) \] on the instability threshold:
  \[ z_m^{TH} + 1 = \frac{1}{\hat{\xi}_{TH}} = \frac{1}{1 - \sqrt{\lambda}} \]

In dimensional magnitudes:

\[ \rho g (\xi_m^{TH} + \xi_0) \left( \frac{1}{\sqrt{3Y}} \right) = 1 \]

- Stability requires that hydrostatic pressure across a crest be less than \( \sqrt{3Y} \).
Analytic solutions

- There are two kinds of unstable solutions. Both with the same asymptotic growth rate $\sqrt{kg}$, typical of an ideal fluid.

- And two kinds of stable solutions. Both have the same period, typical of a perfectly elastic solid:

$$T = \frac{2\pi}{\sqrt{kg}} \frac{\hat{\lambda}}{1 - \hat{\lambda}}; \quad \hat{\lambda} = \frac{\rho g \lambda}{4\pi G}$$

$$z = \frac{\xi}{\xi_0}$$

$$\tau = t (\text{kg})^{1/2}$$

a) $\hat{\lambda} = 0.3, \xi = 0.3$
b) $\hat{\lambda} = 0.3, \xi = 0.4$
c) $\hat{\lambda} = 0.3, \xi = 0.45$
d) $\hat{\lambda} = 0.3, \xi = 0.5$
e) $\hat{\lambda} = 1.5, \xi = 0.3$
Stable solutions:

Dots are the result of extensive 2D numerical simulations with the finite element code ABAQUS.
Implications for the LAPLAS experiment

To assure the stabilization of the shortest wavelengths the initial asymmetry must be below the instability threshold. The cutoff wavelength is:

\[ \frac{\lambda_c}{h} = \frac{4\pi G}{p_0} \left(1 - \frac{p_0}{\sqrt{3Y} p_0} \right)^2, \quad \frac{\Delta p}{p_0} = \frac{\xi_0}{h} \]

With the wobbler we have (Piriz et al. PPCF, 2003):

\[ \frac{\Delta p}{p_0} = \frac{1}{N^2}; \quad N = \frac{\tau_p}{T_w} \]

\[ \tau_p = 100 \text{ ns}; \quad p_0 = 1.5 - 10 \text{ Mbar (FAIR)} \]

\[ T_w \approx 3 \text{ ns} \quad (A. \ Golubeb, \ Hirschegg, \ 2009) \]

\[ \frac{\rho g \xi_0}{\sqrt{3Y}} < 1 \implies Y > \frac{p_0}{\sqrt{3} p_0} \Delta p \approx 0.1 - 0.6 \text{ GPa} \]
### Implications for the LAPLAS experiment

\[ \lambda_c > 0 \Rightarrow Y > \frac{p_0}{\sqrt{3}} \frac{\Delta p}{p_0} \approx 0.1 - 0.6 \text{ GPa} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>Y (GPa)</th>
<th>G (Gpa)</th>
<th>(\rho) (g/cm(^3))</th>
<th>(T_m) (K)</th>
<th>vote</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 W</td>
<td>2.2 – 4.0</td>
<td>160</td>
<td>19.25</td>
<td>4520</td>
<td>Very good</td>
</tr>
<tr>
<td>2 Ta</td>
<td>0.77 – 1.0</td>
<td>70</td>
<td>16.65</td>
<td>4340</td>
<td>Very good</td>
</tr>
<tr>
<td>3 Nb</td>
<td>0.70 – 1.4</td>
<td>85.5</td>
<td>8.57</td>
<td>2330</td>
<td>Very good</td>
</tr>
<tr>
<td>4 Ti</td>
<td>0.71 – 1.45</td>
<td>43.4</td>
<td>4.507</td>
<td>2260</td>
<td>Very good</td>
</tr>
<tr>
<td>5 Stainless Steel</td>
<td>0.34 -2.5</td>
<td>77</td>
<td>7.85</td>
<td>2380</td>
<td>Good</td>
</tr>
<tr>
<td>6 Be</td>
<td>0.33 –1.23</td>
<td>151</td>
<td>1.848</td>
<td>1820</td>
<td>Good</td>
</tr>
<tr>
<td>7 Al</td>
<td>0.29-0.68</td>
<td>27.6</td>
<td>2.7</td>
<td>1220</td>
<td>Not bad</td>
</tr>
<tr>
<td>8 Au</td>
<td>0.02-0.225</td>
<td>28</td>
<td>19.3</td>
<td>1970</td>
<td>Bad</td>
</tr>
<tr>
<td>9 Pb</td>
<td>0.008-0.1</td>
<td>8.6</td>
<td>11.34</td>
<td>760</td>
<td>Very bad</td>
</tr>
</tbody>
</table>

(Steinberg et al., J. Appl. Phys., 1980)
Concluding remarks

- We have constructed a physical model for the RTI in solids. The model is simple and very accurate as it is demonstrated by the excellent agreement with 2D ABAQUS numerical simulations.

- Instability threshold is clearly differentiated from the elastic-plastic transition. EP transition is a necessary conditions but not sufficient for instability. The onset of instability is related to the hydrostatic pressure across a perturbation crest: instability occurs when it overcomes the value $\sqrt{3}Y$.

- The instability threshold is determined by both, the perturbation amplitude and perturbation wavelength. This is a distinctive characteristic of RTI in solids.

- LAPLAS target needs to consider other metals as pusher materials different than Au or Pb in order to make it more robust with respect to the hydrodynamic instabilities.