Frequency map analysis of resonances in a Nonlinear Lattice with Space Charge

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Abstract

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In storage rings for heavy ion fusion beam losses must be minimized. During bunch compression high space charge is reached and the reciprocal effects between the collective modes of the beam and the single particle lattice nonlinearities must be considered to understand the problem of resonance crossing and halo formation. We show that the frequency map analysis of particle in core models gives an adequate description of the resonance network and of the chaotic regions where the halo particles can diffuse.

1 Introduction

The resonances contribute to expand the halo in high current beams through the slow diffusion in their stochastic web1,2. New resonances appear when the beam is mismatched and in presence of lattice nonlinearities. Both effects should be taken into account in storage rings for heavy ions fusion3. We investigate here the transverse dynamics of the beam in a constant focusing (CF) or periodic focusing (FODO) cell4 with a thin sextupole. For a matched beam the most relevant feature is the non monotonic behaviour of the horizontal tune, which slowly decreases inside the core where the sextupole force is weak, rises where the space charge force still dominates and decreases where the sextupole force prevails, see figure 1. The average behaviour of the tune is fairly well described by first order perturbation theory for a CF cell with axisymmetric beam. No relevant changes occur in a FODO cell. Our analysis is based on the particle in core (PC) model with a KV distribution1,2. Its use for moderate space charges (ν/ν0 ≥ 0.8) is justified for a matched beam, if the core radius is much smaller than the dynamic aperture (R/D ≤ 0.1), as we have checked by solving the Poisson-Vlasov (PV) equations. The resonances due to mismatch oscillations5,6 in a linear lattice are detected by the frequency map7,8. The lattice nonlinearities create new resonances and a wide stochastic region around the (1,-1) resonance. Here the particles may diffuse to reach dynamic aperture, where they are very rapidly lost.

2. The model

We consider a cell of length L with a thin sextupole and a beam with equal emittances ε0 on both axis and a pervance ξ. The single particle Hamiltonian reads

\[ H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + k_{0x} \frac{x^2}{2} + k_{0y} \frac{y^2}{2} - \frac{K_2}{6} (x^3 - xy^2) \delta_p (s) + \frac{\xi}{2} V(x,y) \] (1)

where \( \delta_p (s) \) is the periodic Dirac function of period L, V(x,y) is the space charge potential satifying the Poisson equation

\[ \Delta V = -4\pi \rho_s \quad \rho_s (x,y) = \int \rho(x,p_x,y,p_y) \, dp_x \, dp_y \] (2)

1
\[ \frac{\partial \rho}{\partial s} + [\rho, H] = 0 \quad \int \rho \, dx dy dp_x dp_y = 1 \quad (3) \]

If the integrated sextupolar gradient \( K_2 \) vanishes, then for a KV beam \( V \) depends on the amplitudes \( A_x, A_y \) which satisfy the envelope equations. They have a canonical form with Hamiltonian

\[ H_{\text{env}} = \frac{p_{A_x}^2}{2} + \frac{p_{A_y}^2}{2} + k_{0x} \frac{A_x^2}{2} + k_{0y} \frac{A_y^2}{2} + \frac{\xi_0^2}{2A_x^2} + \frac{\xi_0^2}{2A_y^2} - 2\xi \log(A_x + A_y) \quad (4) \]

We shall use the dimensionless variables \( x = (\epsilon_0 L)^{1/2} x' \), \( p_x = (\epsilon_0/L)^{1/2} p'_x \), \( \beta_x = L \beta_x' \), \( A_x = (\epsilon_0 L)^{1/2} A_x' \), \( s = L s' \). \( L \) and \( \omega_x \) be the linear map and horizontal tune within the core and \( L_x = UR(\omega_x)U^{-1} \) where \( U_{11} = U_{22}^{-1} = \beta_x'^{1/2} = A_x' \) and \( U_{12} = -\alpha_x \beta_x^{-1/2} = dA_x'/ds' \), the normalized coordinates are \( (X', P_x') = U^{-1} (x', p'_x) \). If \( \xi = 0 \) and \( K_2 \neq 0 \) the one turn map reduces to the Hénon map after a scaling. We denote by \( D^{\text{Hen}}(\omega) \) its dynamic aperture, defined as the largest abscissa of the points on the axis \( x \geq 0 \) whose orbits are stable and connected. The ratio between the beam amplitude and the dynamic aperture is \( A_x/D_x = K_2 A_x^3/(2 \epsilon_0 D^{\text{Hen}}) \). For \( K_2 = 0.1 \, \text{m}^{-2} \), \( A_x = 1 \, \text{cm} \) and \( 2 \epsilon_0 = 10^{-6} \, \text{m} \), the ratio is 1/10.

### 3. Perturbation theory

To understand the effect of sextupolar and space charge forces we consider an axially symmetric KV beam in a CF channel, assuming the electric field is not affected by the sextupole. This is correct when the ratio of the beam core radius to the dynamic aperture tends to 0. Letting \( \omega_0 = L\sqrt{K_0} \), \( \omega = L\sqrt{K} \) be the bare and depressed frequencies and \( R_x \) be the beam radius determined by \( \epsilon_0^2/R_x^4 = k_0 - \xi/R_x^2 = k \), we scale the transverse coordinates with \( R_x \) rather than \( \sqrt{\epsilon_0 L} \) and the Hamiltonian reads

\[ H' = \frac{p'_{x}^2}{2} + \frac{p'_{y}^2}{2} + \omega^2 x'^2/2 + \omega^2 y'^2/2 - \frac{K_2}{6} (x'^3 - 3x'y'^2)\delta_{\rho}(s') + \frac{\omega_0^2 - \omega^2}{2} [-\log(r'^2) + r'^2 - 1] \partial(r' - R') \quad (5) \]

where \( r' = (x'^2 + y'^2)^{1/2} \) and \( R' = R/R_x \) satisfies the equation

\[ \frac{d^2 R'}{ds'^2} - \frac{\omega^2}{R'^3} + \frac{\omega_0^2 R' - \omega_0^2 - \omega^2}{R'} = 0 \quad (6) \]

The equilibrium solution is \( R' = 1 \) and the small oscillations frequency is \( \omega_{\text{min}}^2 = 2(\omega_0^2 + \omega^2) \). The tune shift behaviour is given by the first order perturbation theory in \( \xi' = \omega_0^2 - \omega^2 \) and \( K_2' \). For a KV beam with low space charge and a small sextupolar strength it is justified to sum up the first order perturbative calculation of the tune shift due to space charge with the tune shift due to the sextupole. The tune shift due to the sextupole reads

\[ \delta \omega_{\text{sex}} = -\frac{1}{16} [3 \cot \frac{\omega}{2} + \cot \frac{3 \omega}{2}] |z - \phi_{20} z^2 - \phi_{11} z z' - \phi_{02} z'^2|^2 z = K_2' / 2 \beta_x'^{3/2} (X' - iP_x') \quad (7) \]

\[ \uparrow \] The Hamiltonians scale as \( H = H' \epsilon_0 L^{-1} \), \( H_{\text{env}} = H'_{\text{env}} \epsilon_0 L^{-1} \), densities according to \( \rho = \rho'/\epsilon_0^2 \), \( \rho_s = \rho_s'/\epsilon_0 L \) whereas the potential remains unchanged \( V = V' \). The equations of motion, the Liouville and Poisson equation are invariant and the new Hamiltonians are just the old ones with primed variables, \( k_{0x}(s') = L^2 k_{0x}(LS') \), \( K_2' = \sqrt{\epsilon_0} L^{3/2} K_2 \), \( \xi' = \xi L/\epsilon_0 \) and \( \epsilon_0 \rightarrow 1 \). If \( \epsilon_0 \neq \epsilon_{0y} \) one defines \( \epsilon_0 = \sqrt{\epsilon_{0x} \epsilon_{0y}} \) and the scaled Hamiltonians depends on the ratio \( \epsilon_{0y}/\epsilon_{0x} \).
where the coefficients $\phi_{ij}$ are given in [9], but we can set them equal to zero without a significant loss of accuracy. The tune shift due to the space charge is given by

$$\delta \omega_{s.c.} = \frac{\omega_0^2 - \omega^2}{\omega \pi} \left[ (1 - 2J^{-1}) \arccos(J^{-1/2}) + J^{-1} \sqrt{J - 1} \right]$$

Asymptotically for $J \to \infty$ one has $\omega + \delta \omega_{s.c.} \to \omega [1 + (\omega_0^2 - \omega^2)/2 \omega^2]$ which is equal to $\omega_0$ up to terms of order $(\omega_0 - \omega)^2$. The tune shift of the sextupole is decreasing whereas the tune shift of the space charge is increasing. Their sum is no longer monotonic and a maximum is reached between the core and the dynamic aperture. In figure 1 we show the phase plots of a test particle affected by space charge, sextupole and compare the tunes with the perturbative calculation.

![Phase plots and comparison of the the tune (red) with the perturbative solution (green) for a test particle moving along the x axis in a axisymmetric CF channel: only space charge (left), only sextupole (middle), both (right). The parameters for the upper figures are: $\nu_0=1.9, \nu=0.18, K_2=0$ (left) $\nu_0=0.18, \nu=0.18, 1/2 K_2=0.2$ (middle) $\nu_0=.19, \nu=0.18, 1/2 K_2=0.2$ (right). Lower figures: $\nu_0=0.21, \nu=0.18947, K_2=0$ (left) $\nu_0=0.18947, \nu=0.18947, 1/2 K_2=0.1$ (middle) and $\nu_0=0.21, \nu=0.18947, 1/2 K_2=0.1$ (right)](image)

4 Frequency map analysis

The map which associates to any orbit its tunes has been used in celestial mechanics [10,11] and beam dynamics [12,13]. In weakly perturbed integrable systems the frequency map is a one to one map between the orbits and the frequencies. The non-resonant invariant tori on which the motion is quasiperiodic, admit a smooth interpolation. Accurate Fourier analysis of the orbits (with a $n^{-4}$ error) allows to detect non-resonant, resonant and chaotic orbits. In the first case the tune varies smoothly, in the second it is locked, in the third it is sensitive to small variations of the initial conditions and of the orbit length. The frequency map allows to detect the resonances arising when a parameter has a periodic or quasi-periodic time dependence. Consider the horizontal motion of a test particle of a mismatched beam in a CF channel, whose radius oscillates with frequency $\omega_{mis} = 2\pi \nu_{mis}$ and let $\Omega = 2\pi \nu$ be the nonlinear betatronic frequency. The resonance condition reads $k_1 \nu + k_2 \nu_{mis} + k_3 = 0$. In the one turn map the main resonant orbit where $\nu_{mis} = 1/2$ is not recognizable. Since the horizontal motion in a matched beam is integrable, the resonance appears by taking a section in the mismatch phase $\phi_2 = \omega_{mis} s' = 0$ mod $2\pi$.

When the sextupole is present the horizontal motion for the matched beam is no longer integrable. As a consequence the mismatch resonance is no longer recognizable on any section unless the stroboscopic map is taken in the very special case when $\nu_{mis}$ is itself rational, as shown by figure 2 where $\nu_{mis} = 2/5$. The tune exhibits the resonance. In the density plot the resonance is a
Figure 2 One turn map for a 10% mismatched beam in CF cell with \( \nu_0=0.21, \nu=0.18947 \) implying \( \omega^{\text{mis}}=2/5 \) and its tune plot (upper left). Map \( \phi_2=\omega^{\text{mis}} s'=0 \mod 2\pi \) and its tune (upper right). Same cell with a sextupole \( K'_2=0.1 \) (bottom): orbits of the stroboscopic 5 turns map (left), one turn map (middle left), one turn map tune plot (middle right) and tune hystogram (right).

peak, surrounded by an empty region, see figure 2. We have considered a FODO cell whose bare tunes are \( \nu_{0x}=0.1785, \nu_{0y}=0.1645 \) and a beam with \( \xi'=0.5 \) so that the depressed tunes are \( \nu_x=0.1438, \nu_y=0.1285 \). For a sextupolar strength \( K'_2=0.1 \) the ratio of the beam radius to dynamic aperture is \( \sim 1/10 \). The mismatch oscillations excite the 1/2 resonance and since \( \nu^{\text{mis}}/2=0.1588 \) it overlaps the 1/6 resonance; oscillation of the betatron tune between these values are observed, see figure 3. New resonances are excited by the sextupole, as shown also by the plot of points in the \( \nu_x, \nu_y \) plane in figure 4. Using colors to label different resonances in the \((x,y)\) plane shows how chaotic regions at the resonance intersection arise. The most dangerous resonance is the (1,-1) and the chaotic region around it is increased by mismatch. The reliability of the PC model in a nonlinear lattice with \( A/D \ll 1 \) has been checked by solving the PV equations, for a matched beam. The macroparticles evolve by steps \( \Delta s \) according to symplectic micro-maps \cite{14} and the electric field is computed from a FFT solution of Poisson’s equation.
The PC-PV discrepancy for a matched beam is small on the tune, see figure 5, not visible on space phase plots. To conclude the PC model is adequate to describe the halo development, given its initial distribution. The frequency map analysis provides a detailed picture of resonances and of chaotic regions. The relevant effect of sextupolar nonlinearities is to bend the horizontal tune, to widen the chaotic region and a to create a fast escape mechanism (beyond the dynamic aperture).

References