Dynamics and thermodynamics of a gas of automata

G. Turchetti$^{1,2}$, F. Zanlungo$^{1,2}$ and B. Giorgini$^1$

$^1$ Dipartimento di Fisica, Università di Bologna - via Irnerio 46, 40126 Bologna
$^2$ Centro interdipartimentale Galvani -via S. Giacomo 12, 40126 Bologna

PACS 89.75.-k – Complex systems
PACS 87.23.Ge – Dynamics of social systems
PACS 05.10.-a – Computational methods in statistical physics and nonlinear dynamics

Abstract. - We consider a system of point charges interacting within a cone of vision and confined by an external potential, as a simple model of individuals provided with vision. The non Newtonian nature of the interaction introduces dissipative effects which are balanced by a memory mechanism. The two body system is amenable to quadrature, whereas the $N \gg 1$ body system exhibits crystal like and disordered states with a non trivial phase diagram if the interaction range and memory persistence are chosen as control parameters.

Introduction. – The interactions based on perception, as vision, do not obey the third principle of Newtonian dynamics. In this paper we study the effect that this non Newtonian behaviour has on the dynamics of a two bodies and a many bodies system.

We propose the gas of Von Neumann automata as a basic model for complex systems formed by a large number of interacting individuals provided with a sensory system, such as a crowd or a swarm [1]. With “gas of automata” we mean a system similar to a gas of charged particles in continuous 2D space confined by an external potential, except for the presence of a perception system, described by a visual cone of range $r_v$ and angular half-aperture $\alpha$. An automaton $A$ is repelled by an automaton $B$ when it falls within its visual cone, and feels no interaction when $B$ is out of the cone (see figure 1). Repulsion decreases with distance and we assume an inverse proportionality as in the 2D electrostatic case. More explicitly the force is given by

$$ F = \begin{cases} \frac{r}{r^2} & \text{if } B \in C_A \\ 0 & \text{if } B \notin C_A \end{cases} \quad (1) $$

where $r = r_A - r_B$ is the displacement and $r = |r|$ is the distance from $A$ to $B$. The cone condition is defined by

$$ B \in C_A \text{ if } r < r_v \text{ and if } \frac{|\theta|}{\cos \theta} = \frac{r \cdot v}{r v} < \alpha \quad (2) $$

where $v$ is the velocity of $A$, along which we choose the axis of the cone (figure 1).

The model is supposed to describe a “level zero approximation” of a low density crowd in which the automata ("pedestrians") tend to avoid each other, even though in an actual crowd the interactions are certainly more involved: social attractive forces, responsible of the formation of small clusters, and more complex behavioural patterns are certainly present in addition to pure misanthropy. The presence of the visual cone renders the force non Newtonian and changes significantly the $N$ automata problem with respect to the $N$ body problem, by introducing a sort of damping (maximised in the $\alpha = \frac{\pi}{2}$ case), since the repulsive automaton to automaton force is mainly opposed to the direction of the motion and thus has negative power.

The model, given the simplicity of both the perception and decision systems of automata, and also the point-like nature of our automata, is not to be intended to be able of describing actual crowd dynamics, but is just a toy model that allows to investigate, using both analytical and numerical methods, the non Newtonian features introduced...
and the other. Nevertheless, since the angular momentum is not conserved in the sharp transitions between one zone the centre of mass motion is cumbersome due to the cone condition (the equations for the cone condition, and the equation of motion can be integrated, even if the integration is quite cumbersome due to the time dependence of the potential). We can write the forces felt by the automata as

\[ F_{12} = -\omega^2 r_1 + \frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}^2} \vartheta(C_1) \]

\[ F_{21} = -\omega^2 r_2 + \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}^2} \vartheta(C_2) \]

where the cone conditions are expressed (assuming for simplicity’s sake \( r_n = \infty \)) by

\[ C_1 = v_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1) - v_1 r_{12} \cos \alpha \]

\[ C_2 = v_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2) - v_2 r_{21} \cos \alpha \]

and \( \vartheta(u) \) is the step function

\[ \vartheta(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ 1 & \text{if } u > 0 \end{cases} \]

We can then pass to the centre of mass \((\mathbf{R}, \mathbf{V})\) and relative distance \((\mathbf{r}, \mathbf{v})\) coordinates, and we obtain three different Hamiltonian functions for the relative motion, depending on the relative positions and velocities of the automata. The Hamiltonian for the relative motion reads

\[ H_I = H \equiv \frac{v^2}{2} + \frac{\omega^2 r^2}{2} \]

when none of the automata sees the other one;

\[ H_{II} = H + V(r) \quad V(r) \equiv -\ln(r) \]

when just one of them sees the other one;

\[ H_{III} = H + 2V(r) \]

where the cone condition is given by \( C \equiv -\dot{r} - L \cot \alpha \). When \( \alpha = \frac{\pi}{2} \) the cone condition simplifies to \( C \equiv -\dot{r} \)

which means that the switch between the two integrals of motion occurs when an inversion point is reached (the automaton feels a repulsive potential only when approaching the origin).

It is easy to show (figure 2) that the automaton reaches a relative equilibrium whenever its inversion point is between the minima of the effective potentials acting when the cone condition is verified or when it is not. To this equilibrium point corresponds a circular orbit for the radial motion. We have verified, using a second order symplectic integrator, that for any value of \( \alpha \) the asymptotic orbits of the centre of mass and of the relative motion are closed curves, with an energy lower than the initial one (see figure 3). An analogous behaviour has been found, always using numerical integration, for the radial motion of the two automata problem.

\[ H = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} + \frac{\omega^2 r^2}{2} - \ln(r) \vartheta(C) \]

The one automaton problem. – The nature of the problem can be better understood studying a single automaton moving in a confining harmonic potential and in a repulsive Coulombic potential that it feels only when the centre of forces falls in its cone of vision. It can be shown that the problem becomes one dimensional and the radial motion is described by the Hamiltonian

\[ H = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} + \frac{\omega^2 r^2}{2} - \ln(r) \]

The N automata problem. – Our numerical study shows that the loss of energy grows with the number \( N \) of automata, leading quickly to a “frozen” state with temperature \( T = 0 \) (we define temperature as the average kinetic energy of automata). The simulations show that the \( N = 2 \) problem is the only one with a finite equilibrium temperature (figure 4). The most natural way to avoid freezing would have been
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Fig. 3: Phase space trajectories for the radial motion of automata with different values of $\alpha$.

![Phase space trajectories for different $\alpha$ values](image)

Fig. 7: State diagrams for the gas of automata. The equilibrium values of $T$ are shown for $\alpha = 0.1\pi$ (left), $\alpha = 0.5\pi$ (centre) and $\alpha = 0.9\pi$ (right). $\tau$ is on the $y$ axis, and $r_v$ on the $x$ axis. Red corresponds to high temperature, violet to a low one, as reported on the colour bar. Notice the “high temperature island” for $\alpha = 0.1\pi$.

![State diagrams for different $\alpha$ values](image)
to introduce some kind of “internal degree of freedom” (and eventually even an internal energy), representing the intention of the automaton to reach a given goal. Nevertheless, given our interest in studying the effects of non Newtonian perception on automaton to automaton interaction, and to compare the results to a known physical systems (Coulomb oscillators, [5]), we decided to represent the tendency of automata to reach their goal (the centre) only through the confining potential, and to avoid freezing by introducing a memory mechanism.

The idea at the base of memory is that an automaton can retain some information about the position of another automaton even after that the latter exits its cone of vision. The most natural assumption would be that the “observer” calculates an approximate trajectory (for example, constant velocity motion) for the observed automaton when it exits the cone of vision. To make the the model computationally less expansive we assumed that the observer can actually know the exact position of the observed automaton for a “memory time” $\tau$ (this choice is almost equivalent to calculating an approximate trajectory for low values of $\tau$).

We thus let automaton $A$ feel a repulsion force from $B$ for a time interval $\tau$ after it escapes from its visual cone (if $r < r_v$, where $r$ is the relative distance between automata). By letting $\tau \to \infty$ we recover (after a transient phase) the case of $N$ interacting charges without any visual cone, where the total energy and the average kinetic energy are preserved.

We have verified that for very low values of $\tau$ the system goes to an ordered state with $T \approx 0$ (while the spatial distribution was disordered for $\tau = 0$, figure 5), and that for higher values of $\tau$ the system can reach a non trivial equilibrium system, in which the temperature is positive but different from that obtained in the conservative case (figure 6).

We have performed a throughout numerical investigation of the time evolution for a system of $N = 100$ automata varying the control parameters $\alpha$, $r_v$ and $\tau$. The initial conditions corresponded to a self consistent charge distribution for Coulomb oscillators [5], a distribution with radius $R = 1.84$ in which the period of oscillation for particles was $t \approx 10$. On the base of these values, to obtain a description of all the features of the system, we studied the following ranges of parameters: $0 \leq \alpha \leq \pi$, $0 < r_v < 4$ and $0 < \tau < 20$.

This dependence on the parameters (the “state equation” of the gas of automata) is shown in figure 7. The equilibrium temperature usually decreases when $r_v$ increases at $\alpha$ and $\tau$ fixed, whereas it grows with $\tau$ when the other parameters are fixed. The first rule can be explained considering that, as we stressed before, the introduction of the non Newtonian effect due to vision always leads to a dissipation of kinetic energy, since the negative power that the automaton feels when approaching another automaton (the force is directed opposite to the velocity) is not completely balanced by the positive power felt when going away. It is clear that the higher is the value of $r_v$, the longer the automata interact and thus dissipate. The second rule is quite obvious since we introduced memory in order to attenuate dissipation.

Regarding the $\alpha$ dependence of temperature we can say that in general, keeping the other parameters fixed, the temperature is maximum in the $\alpha = 0$ and $\alpha = \pi$ cases (i.e., in conservative systems), while it attains a minimum for $\alpha = \pi/2$.

Exceptions to these rules are found for low values of $\alpha$, where we found “islands” of high temperature for certain ranges of $\tau$. This happens when memory allows the automaton to feel (as can be explained on the base of geometric considerations), in certain situations, the repulsive force mainly directed along its velocity (positive power) and thus attenuates the damping process. An analysis of the transition to equilibrium in the system for these particular values of the parameters shows the presence not only of a dissipative transient (as for all the other values) but also of a second transient during which a part of the lost energy is regained (figure 8).
Since our system resembles (and is equal to in the $\alpha = \pi$ and $r_v = \infty$ case) a system of Coulomb oscillators, which is known to assume uniform crystal configurations when frozen to $T = 0$ [6], and since our system naturally freezes during time evolution, we studied also the emergence of these organised structures and verified that they are actually present for low values of $\tau > 0$ and for large enough values of $r_v$ (both memory and a large enough range of vision are necessary for the system to organise, but when $\tau$ is too large the system does not freeze). In particular, since we had noticed that in these structures the automata where roughly located at uniform distances from their first neighbours, we defined a “disorder parameter”

$$\gamma = \frac{\Delta d_f}{d_f}$$

(12)

(where $d_f$ is the distance to the first neighbour and $\Delta d_f$ its mean squared deviation) that goes to zero in case of a perfect uniform crystal. The equilibrium values of $\gamma$ are shown in figure 9, while one of these crystals is shown in figure 10.

Conclusions. – The automata model we presented is intended as a very naive model of people moving in an open space with a single attracting point (the centre of the harmonic potential), which renders the system symmetric by rotation. Obviously both the pure misanthropic behaviour of the automata and the harmonic attraction to the centre are not realistic enough to describe human or animal behaviour, but they allow us to compare to a known conservative model (Coulomb oscillators, see [5]), and focus on the non-Newtonian effects due to sensory perception (vision).

Another possible application of our model is the study of an “avenue”, in which we consider two groups of automata moving in opposite directions and study the minimal request for the emergence of organised patterns [7]. Our preliminary study shows that in presence of a constant force field and a dissipation the two groups reach a constant opposite velocity, in the absence of mutual interactions. The repulsive mutual interaction causes self organisation phenomena with the formation of streams, whose properties depend on the vision parameters.

This work might also be used as the base for a more complex model in which we can introduce more realistic social forces that are attractive over a given social radius [4], and also some heterogeneity by sampling the vision parameters and the mutual force strength, in order to simulate the approach to the equilibrium of a system of individuals with contrasting goals (figure 11 shows a metastable and a stable equilibrium configuration for such a system).

We acknowledge Carlo Benedetti for having performed the numerical study of the problem of automata in the avenue. This work was supported in part by a PRIN grant 2005 of the Italian Ministry of University and Research Dinamica e termodinamica di interazioni a lungo raggio

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Fig. 9: Equilibrium values of $\gamma$ are shown for $\alpha = 0.1\pi$ (left), $\alpha = 0.5\pi$ (centre) and $\alpha = 0.9\pi$ (right). $\tau$ is on the $y$ axis, and $r_v$ on the $x$ axis. Red corresponds to a disordered structure and violet to an ordered one, as reported on the colour bar. Ordered crystals are present for low values of $\tau > 0$ and for large enough values of $r_v$. The structure is almost perfect if $\alpha = 0.5\pi$, while it is very poor for low $\alpha$.

Fig. 10: Equilibrium configuration for $N = 100$, $r_v = \infty$, $\alpha = \pi/2$ and $\tau = 0.1$.

Fig. 11: Metastable (left) and stable (right) $T \approx 0$ configurations for automata split in two groups with different “social radii” $r_s$. Green automata have $r_s = 2$, red ones $r_s = 1.5$. 