Notes on Adiabatic Theory

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0.1 Hamilton-Jacobi equation

The canonical structure of Hamilton equation is strictly connected to geometrical properties of the phase flow: let $\Phi^t(q,p)$ the solution of the canonical equations

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

and let $\gamma(\tau) = (q(\tau), p(\tau))$ any closed curve in the phase space, if we consider the 'tube of phase flux' in the extended phase space $\gamma(\tau, t) = \Phi^t(\gamma(\tau))$ then the 1-differential form

$$\omega = pdq - Hdt$$

is closed on $\gamma(\tau, t)$.

Proof: one has to prove

$$d\omega = dp \wedge dq - dH \wedge dt = 0 \quad (p, q) \in \gamma(\tau, t)$$

a direct calculation gives

$$dq = \frac{\partial q}{\partial \tau} d\tau + \frac{\partial H}{\partial p} dt$$

$$dp = \frac{\partial p}{\partial \tau} d\tau - \frac{\partial H}{\partial q} dt$$

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

Then we have

$$d\omega = \left( \frac{\partial p}{\partial \tau} \frac{\partial H}{\partial p} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial \tau} \right) d\tau \wedge dt - \left( \frac{\partial H}{\partial q} \frac{\partial q}{\partial \tau} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \tau} \right) d\tau \wedge dt$$

and the proposition follows.

Let $(Q, P) = T(q, p)$ a change of variables in the phase space, we say that $T$ is a canonical transformation if it preserves the canonical form of the equation: i.e. in the new variable the system (2) reads

$$\dot{Q} = \frac{\partial H}{\partial P}$$

$$\dot{P} = -\frac{\partial H}{\partial Q}$$

As a consequence of the proposition if one consider the phase flow

$$(Q(q, p, t), P(q, p, t)) = \Phi^t(q, p)$$
as a time dependent change of variables, \( \Phi^t(q, p) \) is a canonical transformation.

Proof: we construct a closed surface \( \Sigma(t) \) in the extended phase space using the tube of flux \( \gamma(\tau, t) \) and two surfaces in the phase space whose boundary is \( \gamma(\tau, 0) \) and \( \gamma(\tau, t) \). Then the closure of the differential form \( \omega \) on the \( \gamma(\tau, t) \) implies

\[
\int_{\Sigma(t)} d\omega = \int_{\partial\gamma(t)} dp \wedge dq - \int_{\partial\gamma(0)} dp \wedge dq = 0
\]

so that

\[
\int_{\partial\gamma(t)} dp \wedge dq = \int_{\partial\gamma(0)} dP \wedge dQ = \int_{\partial\gamma(0)} p \wedge dq \quad \forall \gamma(\tau)
\]

The previous condition implies

\[
dp \wedge dq = dP \wedge dQ
\]

which is equivalent to the preservation of the Poisson brackets and, consequently, of the canonical form of the evolution equation.

The relation between the phase flow and the differential form \( \omega \) is bidirectional: indeed if one considers the vector field associated to the null eigenvalue of the differential 2-form \( d\omega \)

\[
d\omega(q, \dot{p}, 1) = 0
\]

we get the solution

\[
\begin{pmatrix}
0 & I & \partial H/\partial q \\
-I & 0 & \partial H/\partial p \\
\partial H/\partial q & \partial H/\partial p & 0
\end{pmatrix} = \begin{pmatrix}
\partial H/\partial p \\
-\partial H/\partial q \\
1
\end{pmatrix} = 0
\]

(3)

where we set

\[(q, \dot{p}, 1) = (\partial H/\partial p, -\partial H/\partial q, 1)\]

and we recover the canonical equations of motion. Let \( \Phi^t(q, p) \) any phase flow and \( H(q, p) \) an Hamiltonian systems, to compute how the Hamiltonian changes if one performs the change of variables, we use the condition

\[
pdq - Hdt = PdQ - H'dt + dF
\]

(4)

that is equivalent to the request of canonicity. Then using the generating function approach \( F(q, Q, t) \) we get an implicit form

\[
p = \frac{\partial F}{\partial q}
\]

\[
P = -\frac{\partial F}{\partial Q}
\]

for the transformation \( \Phi^t(q, p) \) and the Hamilton-Jacobi equation holds

\[
H'(Q, -\frac{\partial F}{\partial Q}) = H(q, \frac{\partial F}{\partial q}) + \frac{\partial F}{\partial t}
\]

(5)
The Hamilton-Jacobi equation allows to solve the dynamics of Hamiltonian systems looking for a change of variables that reduces the initial Hamiltonian to an integrable form. The integrable Hamiltonian systems are defined according to the Liouville theorem:

Given an Hamiltonian system \( H_0(q, p) \in \mathbb{R}^{2n} \) such that there exist \( n \) independent first integrals of motion \( H_k(q, p) \) \( k = 0, \ldots, n - 1 \) in involution

\[
\{ H_k, H_l \} = 0 \quad \forall k, l
\]

\( \{ F, G \} \) is the usual Poisson bracket operator) with the condition that the \( n \)-dimensional surface

\[
\Sigma_k = \{(q, p)/H_k(q, p) = E_k \quad k = 0, \ldots, n - 1\}
\]

is compact, then it is possible to perform a change of variables \((q, p) \rightarrow (\theta, I)\) (action-angle variable) such that \( H_k(q, p) = H_k(I) \) and the canonical equations has an integrable form

\[
\begin{align*}
\dot{\theta} &= \frac{\partial H_0}{\partial I} = \Omega_0(I) \\
\dot{I} &= 0
\end{align*}
\]

The solutions are quasi-periodic functions on an invariant \( n \)-dimensional torus \( I = \text{const.} \)

\[
\theta(t) = \theta_0 + \Omega_0(I)t \mod 2\pi
\]

The proof starts observing that the surface \( \Sigma_E \) is regular (i.e. the tangent space exists at any point) and that the Hamiltonian fields \((\partial H_k/\partial p, -\partial H_k/\partial q)\) \( k = 0, \ldots, n - 1 \) define a base for the tangent space. Then one proves that the differential form

\[
\omega = pdq
\]

is a closed form on the surface \( \Sigma_E \): if one evaluates the 2-differential form \( d\omega = dp \wedge dq \) on the base vector \((\partial H_k/\partial p, -\partial H_k/\partial q)\) it is easy to verify

\[
d\omega \left( \left( \frac{\partial H_k}{\partial p} - \frac{\partial H_k}{\partial q} \right), \left( \frac{\partial H_h}{\partial p}, -\frac{\partial H_h}{\partial q} \right) \right) = \{ H_k, H_h \} = 0
\]

Then we locally define the function on the phase space

\[
F(q, E) = \int_{\Sigma_E}^q p \, dq
\]

and it is possible to introduce local coordinates \( t_k \) on \( \Sigma_E \)

\[
t_k = \int_{\Sigma_E}^q \frac{\partial p}{\partial E_k} \, dq
\]

The coordinates \((t_0, \ldots, t_{n-1})\) define a univocal application from \( \mathbb{R}^n \) to \( \Sigma_E \) which is compact, then there exist many pre-images of the same point of \( \Sigma_E \) that define a discrete subgroup of \( \mathbb{R}^n \). The dimension of this group has to be \( n \) (otherwise \( \Sigma_E \) cannot be compact) and there exist \( n \)-vector \( T_k \) such that the coordinates
0.2 Slowly Modulated Hamiltonian Systems

In its first formulation the theory of adiabatic invariance considers the dynamics of an Hamiltonian system in the form

\[ E = H_0(p, q, \lambda) \]  

where \( \lambda = \epsilon t \) and \( H \) is periodically dependent on \( \lambda \) with period \( 2\pi \). The Hamiltonian \( H_0(p, q, \lambda) \) is called the \textit{frozen system} when we consider the dynamics for a fixed value of the parameter \( \lambda \). We assume that the phase space of the frozen system

\[ t + \sum n_k T_k \]  

represent the same point on \( \Sigma_E \). To each \( T_k \) we can associate a cycle \( \gamma_k \) on \( \Sigma_E \) that results a \( n \)-dimensional torus. We have the relations

\[ T_{k,h} = \oint_{\gamma_k(E)} \frac{\partial p}{\partial E_h} dq \]  

where \( T_{k,h} \) denotes the \( h \) component of the \( k \) vector. The matrix \( T_{k,h} \) is invertible since the vectors \( T_k \) are independent and we can introduce the angle variables

\[ \theta_h = 2\pi \sum_k T_{h,k}^{-1} t_k = 2\pi \sum_k T_{h,k}^{-1} \oint_{\Sigma_E} \frac{\partial p}{\partial E_k} dq \]  

Remark: the origin of the phase variables is arbitrary. It is convenient to introduce the action variables from the relation

\[ dI_k = \sum_k T_{k,h} \frac{1}{2\pi} dE_h \]  

By definition we have

\[ \theta_h = \oint_{\Sigma_E(I)} \frac{\partial p}{\partial I_h} dq \]  

and

\[ I_k = \frac{1}{2\pi} \oint_{\gamma_k(E)} p dq \]  

The transformation in action angle variables in canonical since it can represented by a generating function

\[ F(q, I) = \oint_{\Sigma_E(I)} p dq \]  

In the new variables we have \( H_h(q, p) = H_h(I) \) and

\[ \frac{\partial H_h}{\partial I_k} = 2\pi T_{h,k}^{-1}(I) = \Omega_{h,k}(I) \]  

The solutions of motion for the Hamiltonian \( H_0(q, p) \) are

\[ \theta_k = \theta_k(0) + \Omega_{0,k} t \]  

and in generic cases, correspond to quasi periodic orbits on the invariant torus \( \Sigma_E \).
system is divided into different regions by separatrix curves and in each region we perform the action-angle change of variables

\[ I = I(E, \lambda) = \frac{1}{2\pi} \oint_{H_0(p,q,\lambda)=E} p(E,q,\lambda) \, dq \]

\[ \theta = \left. \frac{\partial F}{\partial I} \right|_q (q, I, \lambda) \]

(12)

where \( F(I, q, \lambda) \) is the generatrix function computed from the definition

\[ F(q, E(I, \lambda), \lambda) = \int_{H_0=E}^{q} p(q, E, \lambda) \, dq \]

(13)

which depend parametrically from \( \lambda \) which is kept fixed during the integrations. In the definition of \( F(q, I, \lambda) \) one can choose the origin of the angle \( \theta \) in an arbitrary way: i.e. in any section of the phase space. Letting \( H_0 = H_0(I, \lambda) \) we have the frozen frequency

\[ \Omega(I, \lambda) = \frac{dH_0}{dI} \]

Let \( \lambda = \epsilon t \) with \( \epsilon \ll 1 \), so that the \( 1/\epsilon \) is a time scale much longer than the characteristic time scales \( 2\pi/\Omega \) of the frozen system: this means that the orbit is not too close to the separatrix. From the definition (12) we have

\[ \theta = \left. \frac{\partial H_0}{\partial I} \right|_q \int_{H_0=E}^{q} \frac{\partial p}{\partial E} \, dq = \Omega(E, \lambda) \int_{H_0=E}^{q} \frac{dq}{\dot{q}} = \theta_0 + \Omega(E, \lambda)t \]

It is a well known result that according to our assumptions, the action \( I(p, q, \lambda) \) of the frozen system is an adiabatic invariant

\[ |I(p(t), q(t), \epsilon t) - I_0| \leq O(\epsilon) \quad \text{for} \quad t \leq \frac{1}{\epsilon} \]

(14)

where \( (p(t), q(t)) \) is the solution of the Hamiltonian (11) with \( \lambda = \epsilon t \). Moreover if \( H_0 \) is periodically dependent on \( \lambda \), the action \( I \) turns out to be a perpetual adiabatic invariant (V.I.Arnold "Small denominators and problem of stability of motion in Classical and Celestial Mechanics" Russ. Math. Survey 18 n.6, 85-192, (1963)): i.e. the inequality (14) holds for all times. The proof of this result is a consequence of the perturbation theory. According to Hamilton-Jacobi theory, changing to the action-angle variables, we get a new Hamiltonian in the form

\[ H(I, \theta, \lambda) = H_0(I, \lambda) + \epsilon H_1(I, \theta, \lambda) \]

(15)

where \( H_0(I, \lambda) = H_0(p, q, \lambda) \) and we have

\[ \left. \frac{\partial F}{\partial \lambda} \right|_{q, I} (I, \theta, \lambda) = H_1(I, \theta, \lambda) \]

(16)

**Lemma** It is possible to choose \( F(q, I, \lambda) \) in order that

\[ \left. \left\langle \frac{\partial F}{\partial \lambda} \right|_{q, I} (I, \theta, \lambda) \right\rangle = 0 \]

(17)
0.2. SLOWLY MODULATED HAMILTONIAN SYSTEMS

**Proof** From the definition (13) we explicitly compute

\[
\frac{\partial F}{\partial \lambda} \bigg|_{q,I} = \int^q \left( \frac{\partial p}{\partial \lambda} \bigg|_{q,E,\lambda} + \frac{\partial p}{\partial E} \bigg|_{q,\lambda} \frac{\partial H_0}{\partial \lambda} \bigg|_I \right) dq
\]

From the equality

\[H_0(q, p(q, E, \lambda), \lambda) = E\]

we get

\[
\frac{\partial H_0}{\partial p} \bigg|_{q,\lambda} \frac{\partial p}{\partial E} \bigg|_{q,E,\lambda} + \frac{\partial H_0}{\partial \lambda} \bigg|_I = 0
\]

so that we have the equation

\[
\int^q \frac{\partial p}{\partial \lambda} \bigg|_{q,E} dq = -\int^q \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} \left( \frac{\partial H_0}{\partial p} \bigg|_{q,\lambda} \right)^{-1} dq = -\frac{1}{\Omega(E, \lambda)} \int^0 \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} d\theta
\]

Analogously form the equality

\[H_0(I(E, \lambda), \lambda) = E\]

we get

\[
\frac{\partial H_0}{\partial I} \bigg|_{\lambda} \frac{\partial I}{\partial E} \bigg|_{E,\lambda} + \frac{\partial H_0}{\partial \lambda} \bigg|_I = 0
\]

Therefore the following equation holds

\[
\int^q \frac{\partial p}{\partial E} \bigg|_{q,E,\lambda} \frac{\partial H_0}{\partial \lambda} \bigg|_I dq = -\int^0 \frac{\partial I}{\partial \lambda} \bigg|_{E} d\theta
\]

Recalling the definition (12) we compute

\[
\frac{\partial I}{\partial \lambda} \bigg|_E = \frac{1}{2\pi} \int_{H_0=H} \frac{\partial p}{\partial \lambda} \bigg|_{q,E} dq
\]

and we prove that

\[
\frac{\partial I}{\partial \lambda} \bigg|_E = -\frac{1}{2\pi \Omega(E, \lambda)} \int^2 \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} d\theta = -\frac{1}{\Omega(E, \lambda)} \left< \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} \right>
\]

Finally we have the relation

\[
\frac{\partial F}{\partial \lambda} \bigg|_{q,I} = -\frac{1}{\Omega(E, \lambda)} \int^0 \left( \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} - \left< \frac{\partial H_0}{\partial \lambda} \bigg|_{q,p} \right> \right) d\theta
\]

therefore integrating on a period \([0, 2\pi]\) we get the Lemma (17).

**Remark:** \(F(\theta, I, \lambda)\) diverges when one approaches the separatrix.

Keeping \(\lambda\) constant, we can apply the perturbation theory to compute an improved invariant \(J\) (associated to an angle variable \(\phi\)) such that the Hamiltonian (15) is transformed according to

\[H'(J, \phi, \lambda) = H_0(J, \lambda) + O(\epsilon^2)\]  (18)
**Proof:** we proceed with a perturbative expansion by applying a canonical transformation associated to the generating function

\[ G(\theta, J, \lambda) = J\theta + \epsilon G(\theta, J, \lambda) \]

according to

\[ \theta = \phi - \epsilon \frac{\partial G}{\partial J}(\phi, J, \lambda) + O(\epsilon^2) \]

\[ I = J + \epsilon \frac{\partial G}{\partial \phi}(\phi, J, \lambda) + O(\epsilon^2) \]

and we have the equation

\[ H_0(J, \lambda) + \epsilon \frac{\partial H_0}{\partial J} \frac{\partial G}{\partial \phi} + \epsilon \frac{\partial F}{\partial \lambda} q,I (I, \theta) + O(\epsilon^2) = H_0(J, \lambda) + O(\epsilon^2) \]

where we have used the definition (16). The function \( G(\phi, J) \) satisfies the homological equation

\[ -\Omega(J, \lambda) \frac{\partial G}{\partial \phi} = \frac{\partial F}{\partial \lambda} q,I (\phi, J, \lambda) \]

The solution exists since

\[ \left\langle \frac{\partial F}{\partial \lambda} q,I (\theta, I, \lambda) \right\rangle = 0 \]  

(20)

and we have

\[ G(\phi, J, \lambda) = -\frac{1}{\Omega(J, \lambda)} \int_{\phi}^{\lambda} \frac{\partial F}{\partial \lambda} q,I (\phi, J, \lambda) d\phi \]

From the definition (19) we have the relation

\[ I = J + O(\epsilon^2) \]  

(21)

and the (18) is satisfied.

We remark as the previous calculations to get the relation (16) apply also to a modulated map case when we introduce the interpolating Hamiltonian.

The adiabatic invariance of the action variable follows directly from the inequality

\[ |J(t) - J(0)| < O(\epsilon) \quad \text{for} \quad t < \frac{1}{\epsilon} \]

and the relation (21). As a byproduct we can prove that the new action \( J \) is an improved adiabatic invariant. Indeed using the generating function \( G(\theta, J, \lambda) \) with \( \lambda = \epsilon t \) we have a new Hamiltonian in the form

\[ H'(J, \lambda) = H_0(J, \lambda) + \epsilon^2 \frac{\partial G}{\partial \lambda}(J, \phi, \lambda) + O(\epsilon^3) \]
and by repeating the previous procedure one can prove that

\[ |J(q(t), p(t), \epsilon_t) - J_0| \leq O(\epsilon^2) \quad \text{if} \quad t \leq \frac{1}{\epsilon} \]

In the 1D case for Hamiltonian systems it is possible to prove the existence of a perpetual adiabatic invariant (not too close to a separatrix) using KAM theory (V.I. Arnold).

For seek of completeness we can obtain a proof of the inequality (14) directly from the equation

\[
\frac{dI}{dt} = \epsilon \left( \frac{\partial I}{\partial E} \frac{\partial H_0}{\partial \lambda} \bigg|_{p,q} + \frac{\partial I}{\partial \lambda} \frac{\partial H_0}{\partial E} \bigg) \right.
\]

which can be written

\[
\frac{dI}{dt} = \frac{\epsilon}{\Omega(E, \lambda)} \left( \frac{\partial H_0}{\partial \lambda} \bigg|_{p,q} - \left( \frac{\partial H_0}{\partial \lambda} \right)_{p,q} \right)
\]

In the case of modulated maps, the previous relation is substituted by a discrete relation for the evolution of the action \( I \). Let us choose a section \( \theta = \text{const.} \) in the phase space (Poincaré section) and let \( \{t_n\}_{n \in \mathbb{N}} \), the sequence of crossing times of an orbit \((q(t), p(t))\) with the section. Since the frozen energy \( H_0(p, q, \lambda) \) is conserved up to an error of order \( O(\epsilon) \), the solution \((q(t), p(t))\) for \( t \in [t_n, t_{n+1}] \) is close to the curve \( H_0(p, q, \lambda_*) = E_* \) where \( \lambda_* \in [\lambda_n, \lambda_{n+1}] \) and \( E_* \in [E_n, E_{n+1}] \). Moreover the dynamics of the angle variables reads

\[
\dot{\theta} = \Omega(E, \lambda) + \epsilon \frac{\partial H_1}{\partial I}
\]

Then we get the relation

\[
dt = - \frac{d\theta}{\Omega(E, \lambda)} + O(\epsilon)
\]

and we can use \( \theta \) as a parameter for the dynamics \((q(t), p(t))\). According to the previous definition we have

\[
I_{n+1} - I_n = \epsilon \int_0^{2\pi} \frac{1}{\Omega^2(E, \lambda)} \left( \frac{\partial H_0}{\partial \lambda} \bigg|_{p,q} - \left( \frac{\partial H_0}{\partial \lambda} \right)_{p,q} \right) \, d\theta + O(\epsilon^2)
\]

Since both the energy \( E \) and the parameter \( \lambda \) can be kept constant during the integration up to an error of order \( O(\epsilon) \), the previous relation simplifies since \( \Omega(E, \lambda) \) can be put out of the integral

\[
I_{n+1} - I_n = O(\epsilon^2)
\]

As a consequence the value of \( I \) cannot vary more than a quantity of order \( O(\epsilon^2) \) in the interval \([t_n, t_{n+1}]\) and the adiabatic invariance holds.

Finally we remark that the estimate requires that \( \Omega(E, \lambda) \) is not too close to 0: i.e. the orbits are not too close to a separatrix where the unperturbed frequency \( \Omega(E, \lambda) \to 0 \) as \( 1/\ln |E - E_*| \) \( (E_* \) is the separatrix energy). The ratio \( \epsilon = \epsilon/\Omega^2(E, \lambda) \) defines the adiabatic parameter that has to be small for a correct application of a perturbative approach.
0.3 Adiabatic forced harmonic oscillator

We have shown that the problem of adiabatic invariance reduces to the study of the Hamiltonian system (15). To understand the limit of adiabatic invariance due to the non-linear resonances, in a simplified approach we consider the Hamiltonian

$$H(I, \theta, \lambda) = H_0(I, \lambda) + \epsilon \frac{h_k(I)}{k} \sin(k\theta - \alpha t)$$  \hspace{1cm} (25)

that corresponds to a forced Hamiltonian system with a non-linear resonance condition

$$k \frac{\partial H_0}{\partial I} - \alpha = 0$$

We assume that \( \lambda = \epsilon t \) and that the resonance condition is crossed at a certain value \( \lambda_* (I) \) for a given value of the action variable \( I \). It is convenient to introduce the moving frame \( \phi = \theta - \frac{\alpha}{kt} \) using the generating function

$$F(J, \theta) = J \left( \theta - \frac{\alpha}{k} t \right)$$

and the new Hamiltonian reads

$$H(J, \phi, \lambda) = \frac{(kH_0(J, \lambda) - \alpha J)}{k} + \epsilon \frac{h_k(J)}{k} \sin(k\phi)$$

Then we scale the time \( t' = t/k \) and consequently \( H' = kH \) and we get

$$H'(J, \phi, \lambda) = kH_0(J, \lambda) - \alpha J + \epsilon h_k(J) \sin(k\phi)$$  \hspace{1cm} (26)

An approximate solution for the phase \( \phi \) is

$$\phi(t) = \phi_0 + \int_0^t k\Omega_0(J_0, \lambda)ds - \alpha t + O(\epsilon)t$$

so that we compute the evolution of the action variable

$$J(t) = J_0 - \epsilon kh_k(J_0) \int_0^t \cos(\phi_0 + k\Omega_0(J, \lambda)s - \alpha s)ds + ......$$  \hspace{1cm} (27)

Remark: we approximate \( J = J_0 \) during the evolution, that gives an error of order \( O(\epsilon) \) when adiabatic theory applies. Any small changes in the action variable implies a change of order \( O(1) \) in the angle variable that depends quite sensitive from the initial condition at a time scale \( O(\epsilon^{-1}) \). However near the resonance the phase value freezes and we have changes of order \( O(\epsilon) \) per unit time intervals.

As long as \( |k\Omega_0(J_0, \lambda) - \alpha| \gg \epsilon \) we apply the adiabatic theory and the action \( J \) varies by a quantity of order \( O(\epsilon) \). Then we define the crossing parameter \( \lambda_* \)

$$\Omega_0(J_0, \lambda_*) = \frac{\alpha}{k}$$  \hspace{1cm} (28)

When \( \lambda \simeq \lambda_* \) we cannot apply the adiabatic theory and the change of the action variable has to be computed from the integral (27) estimating the remainder. At the resonance crossing we approximate the frequency

$$k \frac{\partial \Omega_0}{\partial \lambda} (J_0, \lambda_*) (\lambda - \lambda_*) + k \frac{\partial \Omega_0}{\partial J} (J_0, \lambda) (J - J_0) + ......$$
and we cross transversely the resonance if the quantity
\[
\frac{\partial \Omega_0}{\partial \lambda}(J_0, \lambda_*) - \frac{\partial \Omega_0}{\partial J}(J_0, \lambda_*) h_k(J_0) \cos(k\phi(t_*)) = \Delta(J_0, \lambda_*) = O(1)
\]
Assuming the previous generic estimate, the phase evolution near the resonance crossing can be approximated by
\[
\phi(t) \simeq \phi_* + k \int_{\lambda_*}^{\lambda} \left[ \frac{\Omega_0}{\partial \lambda}(J_0, \lambda_*) - \frac{\partial \Omega_0}{\partial J}(J_0, \lambda_*) h_k(J_0) \cos(k\phi(t_*)) \right] (\lambda - \lambda_*) d\lambda
\]
plus higher order terms. Then we get an estimate for the action variation from eq. (27)
\[
J(t) = J_0 - \epsilon k h_k(J_0) \int_{t_* - t_0}^{t_* + t} \cos \left( \phi_* + \epsilon \Delta(J_0, \lambda_*) \frac{(s - t_*)^2}{2} \right) ds
\]
By scaling time \( u = (t - t_*) \sqrt{\epsilon \Delta(J_0, \lambda_*)} \), we get
\[
J(t) = J_0 - k \sqrt{\frac{\epsilon}{\Delta(J_0, \lambda_*)}} h_k(J_0) \int_{t_0/\sqrt{\epsilon \Delta(J_0, \lambda_*)}}^{t_0/\sqrt{\epsilon \Delta(J_0, \lambda_*)}} \cos \left( \phi_* + \frac{u^2}{2} \right) du
\]
The integral is a Fresnel integral whose contribution is of order \( O(1) \) according to our assumptions if \( \epsilon \) is sufficiently small. Then we get an estimate for the change of action due to resonance crossing that reads
\[
J(t) = J_0 + O \left( \sqrt{\frac{\epsilon}{\Delta(J_0, \lambda_*)}} \right) \| k \| \| h_k(J_0) \| |k| (29)
\]
The effective change depends on the crossing angle \( \phi_* \) that behaves like a random variable since its value is quite sensitive to the initial conditions. Then the action change due to resonant crossing is well described by a random kick with zero mean value and a variance defined by eq. (29). We finally remark that for analytic function the Fourier component contribution \( \| h_k(J_0) \| \) is exponentially small in the resonant order \( k \) and that a cut off of small terms can be introduced. A simpler case is the slowly modulated forced harmonic oscillator
\[
H = \frac{p^2}{2} + \omega^2(\lambda) \frac{q^2}{2} + q \epsilon \cos \alpha t
\]
where \( \lambda = \epsilon t \) is the slowly varied parameter. For fixed \( \lambda \) (frozen Hamiltonian) we introduce the action-angle variables (the forcing strength is set at \( \epsilon \) to simulate the effect of a non linear resonance)
\[
I = \frac{1}{2\pi} \int_{H=E_0} pdq = \frac{E_0}{\omega(\lambda)}
\]
and the angle \( \theta \) from the relations
\[
q = \sqrt{\frac{2I}{\omega}} \sin \theta
\]
\[
p = \sqrt{2I \omega} \cos \theta
\]
To apply the Hamilton-Jacobi equation we compute the generating function $F(q, I, \lambda)$

$$F(q, I, \lambda) = \int_{H=E_0}^{q} pdq = \int_{H=E_0}^{q} \sqrt{2E_0 - \omega^2 q^2} dq$$  \hspace{1cm} (31)

After some computations we get

$$F(q, I, \lambda) = I \left( \sqrt{\omega^2 Iq - \omega^2} + \arcsin \left( \frac{\omega}{\sqrt{2} Iq} \right) \right)$$

and

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial q} \frac{dq}{d\lambda} = \frac{pq}{\omega}$$

Then the new Hamiltonian reads

$$H = I\omega(\lambda) + \epsilon \sqrt{\frac{2I}{\omega}(\lambda)} \sin \theta \cos \alpha t + \frac{\epsilon I}{\omega(\lambda)} \frac{d\omega}{d\lambda} \sin \theta \cos \theta$$  \hspace{1cm} (32)

To simplify the calculation we consider the effect of a single forcing frequency

$$H = I\omega(\lambda) + \epsilon \sqrt{2I} \sin(\theta - \alpha t) + \frac{\epsilon I}{\omega(\lambda)} \frac{d\omega}{d\lambda} \sin \theta \cos \theta$$  \hspace{1cm} (33)

Then we introduce the moving frame

$$\phi = \theta - \alpha t$$

and the Hamiltonian takes the form

$$H = I(\omega(\lambda) - \alpha) + \epsilon \sqrt{\frac{2I}{\omega(\lambda)}} \sin \phi + \frac{\epsilon I}{\omega(\lambda)} \frac{d\omega}{d\lambda} \sin(\phi + \alpha t) \cos(\phi + \alpha t)$$  \hspace{1cm} (34)

Let $\epsilon = 0$ (frozen system) use the variables

$$Q = \sqrt{2I} \sin \phi$$
$$P = \sqrt{2I} \cos \phi$$

then the frozen system reads

$$H = \frac{P^2 + Q^2}{2}(\omega - \alpha) + \epsilon \frac{Q}{\sqrt{\omega}}$$  \hspace{1cm} (35)

The system reduces to a usual harmonic oscillator if we translate the $Q$ variable

$$Q' = Q - \frac{\epsilon}{(\omega - \alpha)\sqrt{\omega}}$$

We are interested in studying the passage through the resonance $\omega = \alpha$, then the contribution of the Fourier components $\phi \pm \alpha t$ can be averaged. We explicitly consider the dynamics (35): let $\omega - \alpha = \lambda = 2\epsilon t - 1$, with $t \in [0, 1/\epsilon]$, we have the equation

$$\dot{Q} = (2\epsilon t - 1)P$$
$$\dot{P} = -(2\epsilon t - 1)Q - \frac{\epsilon}{\sqrt{\omega}}$$
and we crossed a resonance condition at $t = 1/(2\epsilon)$. It is convenient to translate the $Q$ variable $Q \to Q + \epsilon/\sqrt{\omega}$ so that the second equation reads

$$\dot{P} = -(2\epsilon t - 1)Q + \frac{\epsilon}{\sqrt{\omega}} (2\epsilon t - 1) - \frac{\epsilon}{\sqrt{\omega}} = -(2\epsilon t - 1)Q + \frac{2\epsilon^2}{\sqrt{\omega}} t$$

Remark: the time dependence of $\sqrt{\omega(\Lambda)}$ introduces higher order term.

Then we solve explicitly the linear system

$$\begin{pmatrix} Q \\ P \end{pmatrix} = R(\Lambda(t)) \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + R(\Lambda(t)) \int_0^{1/\epsilon} R(-\Lambda(s)) \begin{pmatrix} 0 \\ 0/2\epsilon^2 t/\sqrt{\omega} \end{pmatrix} ds \quad (36)$$

where we set

$$\Lambda(t) = \int^t 2\epsilon t - 1 dt = \epsilon t^2 - t$$

Taking into account that the integral

$$\int_0^{1/\epsilon} \sin(\epsilon s^2 - s) (2\epsilon s - 1) ds = -\cos (\epsilon s^2 - s)|_0^{1/\epsilon} = 0$$

(in general we get a contribution of order $O(\epsilon)$) the expression (36) reduces

$$\begin{pmatrix} Q \\ P \end{pmatrix} = R(\Lambda(t)) \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + R(\Lambda(t)) \int_0^{1/\epsilon} R(-\Lambda(s)) \begin{pmatrix} 0 \\ 0/\epsilon/\sqrt{\omega} \end{pmatrix} ds \quad (37)$$

and an explicit implies the calculation of integrals of the form

$$\frac{\epsilon}{\sqrt{\omega}} \int_0^{1/\epsilon} \sin(\epsilon t^2) dt$$

which reduce to Fresnel integral after changing variable $u = \sqrt{\epsilon} t$

$$\frac{\sqrt{\epsilon}}{\omega} \int_0^{1/\sqrt{\epsilon}} \sin u^2 du = O(\sqrt{\epsilon})$$

The initial action is simply the area of the circle $P_0^2 + Q_0^2 = 2E$; the rotation $R(\Lambda(t))$ preserves the action values, therefore the ‘error’ is due to the Fresnel integral and it results of order $O(\sqrt{\epsilon})$. This estimate cannot be improved in generic cases.